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# **A hidden variables model for non-relativistic quantum mechanics in terms of probabilities of particle paths.**

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## **Abstract**

It is proved that in non-relativistic quantum mechanics (without spin) the transition probability may be described in terms of particle paths, every path having a (positive) probability. This leads to a stochastic hidden variables theory providing an intuitive picture of particle motion. The change of velocity at every time has a probability that depends on the potential over a large region, at a difference with the local action of classical dynamics. Thus the hidden variables theory is non-local like Bohm's, but not deterministic.

## I. Introduction

In quantum mechanics the state of a physical system is represented by the wavefunction (more generally the statevector). Usually from the wavefunction we cannot predict the actual outcomes of the experiments but only the probabilities of several possible results. Since the early days of quantum mechanics a controversy has existed about the correct explanation of this fact. Indeed this was the main subject of the celebrated debate between Einstein and Bohr, that culminated with the EPR paper<sup>1</sup> and Bohr's prompt reply.<sup>2</sup> Einstein et al. supported the view that the statistical character of the quantum predictions is due to the fact that the wavefunction should be associated to an ensemble of states, that would represent the actual state of the physical system, incompletely known. In contrast Bohr and his followers supported the completeness of quantum mechanics, that is the opinion that the wavefunction gives a complete description of the state. If we accept completeness then we should assume that the laws of physics are not strictly causal, something that Einstein disliked. In contrast the incompleteness hypothesis suggests the possibility of introducing additional parameters in order to define the state of the physical system in more detail.<sup>6</sup> Those parameters have been known as "hidden variables". The mainstream of the scientific community did not support hidden variables, viewed as useless or even impossible. The latter belief was reinforced by the von Neumann theorem of 1932 against hidden variables. However 20 years later David Bohm<sup>3</sup> proposed a specific hidden variables model, recovering old ideas of de Broglie (the particle is guided by a wave) and Madelung (the hydrodynamical model of Schrödinger's equation). Bohm's model proved that hidden variables are indeed possible, which showed that von Neumann theorem rested upon too restrictive assumptions. The subject was clarified in two celebrated papers by John Bell,<sup>4,5</sup> In any case the debate about hidden variables is not just an old historical fact, but an alive subject as shown by the effort made to refute empirically at least a class of hidden variables, namely those local, something achieved only recently,<sup>7,8,9</sup> In this paper I propose a new, nonlocal, hidden variables theory.

In Bohm's model every particle has an actual path which is completely determined by a guiding complex field  $\psi(\mathbf{r}, t)$  once the initial position is fixed. The field  $\psi$  evolves according to Schrödinger equation, that is for a

single particle,

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi \equiv \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \right] \psi, \quad (1)$$

$\nabla^2$  being the Laplacian operator and  $V(\mathbf{r})$  the potential. Eq.(1) may be separated in two real fields  $R(\mathbf{r}, t)$  and  $S(\mathbf{r}, t)$ , that is

$$\psi = R \exp iS/\hbar, \quad (2)$$

whence eq.(1) becomes a couple of real equations. A hidden variables model is obtained by introducing Bohm's assumption that the path,  $\mathbf{x}(t)$ , of the actual particle is determined by the "guiding" relation

$$\frac{d\mathbf{x}(t)}{dt} = \frac{1}{m} \nabla S(\mathbf{r}, t) |_{\mathbf{r}=\mathbf{x}}. \quad (3)$$

Several authors have developped Bohm's model in different directions.<sup>10</sup> Asides from applications (e. g. description of chemical reactions in terms of molecular paths) some authors consider the model a good picture of the quantum behaviour. However there are difficulties for that interpretation, mainly the following:

1) The model is deterministic. This fact may be a good feature for some people, but it is unable to explain the probabilistic character of quantum mechanics. Thus one is compelled to introduce Born's rule as an additional assumption. Indeed there is no reason to use a statistical ensemble when there is a single particle and the motion is deterministic. Also there is no clue as to why the real part of the field should be the (square root of) the probability density in the ensemble.

2) The particle's law of motion appears as a modification of classical mechanics by the addition of a kind of force deriving from a "quantum potential". That potential has the strange feature that the force depends on the gradient of the logarithm of the field  $R(\mathbf{r}, t)$ , whence it may be large even in regions where that field intensity is very small.

3) In any stationary state, where the wavefunction  $\psi$  may be taken as real and therefore  $S = 0$ , the particle is at rest according to eq.(3). Nevertheless it is assumed that the position is random with a probability density determined by the field  $R(\mathbf{r})$ . This looks somewhat inconsistent.

In my opinion these difficulties prevent us to consider Bohm's model as an appropriate picture for the quantum behaviour. In this paper I propose

a different hidden variables model that eliminates the difficulties, I hope. It rests on Feynman's path integral formulation of quantum mechanics. In our formalism the central role is played by the transition probability, rather than the transition amplitude. It is true that going from the amplitude to the probability some information is lost, but still some interesting quantum properties may be calculated, e. g. scattering cross sections. However I leave the study of applications for future articles.

## II. Transition probability in terms of particle paths

Feynman's paths integrals have been very important both from the fundamental and from the practical points of view.<sup>11</sup> But the formalism does not provide an intuitive picture of the quantum evolution. It is the purpose of this paper to show that in nonrelativistic quantum mechanics of particles without spin (QM in the following) it is possible to get from Feynman's formalism an approach where every particle follows a path and there is a *positive* probability for every possible path.

The fundamental solution of Schrödinger eq.(1) is obtained solving it with the initial condition

$$\psi(\mathbf{r}, t_0) = \delta^3(\mathbf{r} - \mathbf{r}_0), \quad (4)$$

where  $\delta^3(\mathbf{r} - \mathbf{r}_0)$  is a three-dimensional Dirac delta. The solution provides a kernel  $A(\mathbf{r}_0, t_0 \rightarrow \mathbf{r}, t)$ , such that the solution at time  $t$  with a general initial condition  $\psi_0(\mathbf{x}, t_0)$  may be obtained via

$$\psi(\mathbf{r}, t) = \int d\mathbf{x} \psi_0(\mathbf{x}, t_0) A(\mathbf{x}, t_0 \rightarrow \mathbf{r}, t), \quad (5)$$

where the integral is understood to be 3-dimensional. Thus any formalism that allows getting the kernel (or transition amplitude)  $A$  may be taken as the basis for QM.

Feynman<sup>11</sup> showed that an appropriate kernel (or transition amplitude),  $A_F(x_a, t_a \rightarrow x_b, t_b)$ , for Schrödinger eq.(1) is the following

$$A_F = \lim_{\varepsilon \rightarrow 0} \left( \frac{1}{2\pi i \varepsilon} \right)^{n/2} \int dx_{n-1} \dots \int dx_1 \prod_{j=1}^{n-1} \exp(-\gamma x_j^2), \quad (6)$$

$$\prod_{j=1}^n \exp \left\{ \frac{i}{2\varepsilon} (x_j - x_{j-1})^2 - \frac{i\varepsilon}{\hbar} V\left(\frac{x_j + x_{j-1}}{2}\right) \right\},$$

where  $\int dx_j$  are integrals over the whole real line and  $\gamma > 0$  is a parameter introduced in order to regularize the integrals. The time interval  $\varepsilon \equiv t_j - t_{j-1}$ , is independent of  $j$  and  $x_0 \equiv x_a, x_n \equiv x_b$ . The limit  $\varepsilon \rightarrow 0$  should be understood with  $n \rightarrow \infty$  fulfilling

$$n\varepsilon = t_b - t_a. \quad (7)$$

From now I will work in one dimension and use units such that  $\hbar = m = 1$ .

Actually other kernels different from  $A_F$  eq.(6) may be used. They would differ from Feynman's by terms of order  $O(\varepsilon)$ . In this paper I will start using another one substituting  $[V(x_{j-1}) + V(x_j)]/2$  for  $[V(x_{j-1} + x_j)/2]$  or, what is equivalent to order  $\varepsilon$ , writing the amplitude in the form

$$\begin{aligned} A(a \rightarrow b) &= \lim_{\varepsilon \rightarrow 0} \left( \frac{1}{2\pi i \varepsilon} \right)^{n/2} \int dx_{n-1} \dots \int dx_1 \prod_{j=1}^{n-1} \exp(-\gamma x_j^2) \\ &\times \prod_{j=1}^n \exp \left\{ \frac{i}{2\varepsilon} (x_j - x_{j-1})^2 - i\varepsilon V(x_j) \right\}. \end{aligned} \quad (8)$$

I omit the proof that eq.(8) leads to the correct probability amplitude, solution of Schrödinger eq.(1), which is almost identical to the one for eq.(6).<sup>11</sup>

In QM the transition probability is the square modulus of the transition amplitude whence we get, taking eq.(8) into account,

$$\begin{aligned} P(a \rightarrow b) &\equiv P(z_a, t_a \rightarrow z_b, t_b) = |A(a \rightarrow b)|^2 \\ &= \lim_{\varepsilon \rightarrow 0} \left( \frac{1}{2\pi i \varepsilon} \right)^n \prod_{j=1}^{n-1} \int dx_j \int dy_j \prod_{j=1}^{n-1} \exp(-\gamma x_j^2 - \gamma y_j^2) \\ &\times \prod_{j=1}^n \exp \left\{ \frac{i}{2\varepsilon} [(y_j - y_{j-1})^2 - (x_j - x_{j-1})^2] \right\} \\ &\times \prod_{j=1}^{n-1} \exp \{ i\varepsilon [V(x_j) - V(y_j)] \}. \end{aligned} \quad (9)$$

and we have taken into account that  $x_0 = y_0 = z_b, x_n = y_n = z_b$ . The quantity  $P(a \rightarrow b)$  has dimensions of probability per square volume and it should be interpreted as a relative probability.

Now I make a change of variables, that is

$$z_j = \frac{1}{2} (x_j + y_j), u_j = x_j - y_j, 0 \leq j \leq n, \quad (10)$$

whence eq.(??) becomes, reordering the integrals and the exponentials,

$$P(a \rightarrow b) = \lim_{\varepsilon \rightarrow 0} \left( \frac{1}{2\pi\varepsilon} \right)^n \prod_{j=1}^{n-1} \int dz_j \int du_j \exp [-\gamma z_j^2 - \gamma u_j^2] \\ \times \exp \left\{ -iu_j s_j + i\varepsilon \left[ V(z_j - \frac{1}{2}u_j) - V(z_j + \frac{1}{2}u_j) \right] \right\}, \quad (11)$$

where  $z_0 = z_a$ ,  $z_n = z_b$  and  $u_0 = u_n = 0$  and

$$s_j \equiv \frac{z_{j+1} - 2z_j + z_{j-1}}{\varepsilon}, j = 1, 2, \dots, n-1. \quad (12)$$

Eq.(11) may be written

$$P(a \rightarrow b) = \frac{1}{2\pi(t_b - t_a)} \lim_{\varepsilon \rightarrow 0} \int dz_1 \dots \int dz_{n-1} \prod_{j=1}^{n-1} Q_j, \quad (13)$$

where

$$Q_j = (2\pi\varepsilon)^{-1} \exp [-\gamma z_j^2] \int du_j \exp [-\gamma u_j^2 - iu_j \cdot s_j] \\ \times \exp \left\{ i\varepsilon \left[ V(z_j - \frac{1}{2}u_j) - V(z_j + \frac{1}{2}u_j) \right] \right\}. \quad (14)$$

The convergence factors  $\exp [-\gamma z_j^2 - \gamma u_j^2]$  derive from the choice of regularization made in eq.(11). Indeed the choice has the virtue of simplicity, but it is not appropriate for our purposes. We need a more slow regularization factor and I will replace eq.(14) by the following one

$$Q_j = (2\pi\varepsilon)^{-1} \exp [-\gamma |z_j|] \int du_j \exp [-\gamma |u_j| - iu_j \cdot s_j] \\ \times \exp \left\{ i\varepsilon \left[ V(z_j - \frac{1}{2}u_j) - V(z_j + \frac{1}{2}u_j) \right] \right\}. \quad (15)$$

Thus eq.(13) gives the quantum transition probability,  $P(z_a, t_a \rightarrow z_b, t_b)$ , in the form of a path integral, every path corresponding to one possible motion of the particle starting in  $(z_a, t_a)$  and finishing in  $(z_b, t_b)$ . For any finite value of  $n$  (and  $\varepsilon = (t_b - t_a) / n$ ) every path is defined by  $n+1$  spacetime

points  $\{z_j, t_j\}$ , with the assumption that the motion in every time interval  $\{j\varepsilon, (j+1)\varepsilon\}$  is uniform. The “weight”

$$W_n(\{z_j\}) \equiv n \prod_{j=1}^{n-1} Q_j, \quad (16)$$

plays the role of the (relative) probability of the path *provided that*  $W_n \geq 0$ . It is trivial to show that the quantity  $W_n$  is real. In fact any  $Q_j$ , eq.(14), is real because the imaginary part of the integrand does not contribute, it having the wrong symmetry. However there may be paths whose “weight”  $W_n\{z_j\}$  is negative for some potentials  $V(x)$ . In the following I show that for a restricted class of potentials weight of every path is positive (or zero) provided that the path is defined with a large enough  $n$ . Nevertheless the restricted class of potentials cover all those physically sensible, as shown in the following.

### III. Positivity of the path weights

The proof that the weights eq.(16) are positive rests upon the following

**Theorem 1** *If the Fourier transform of the potential  $V(x)$  has compact support and its integral over its domain is bounded, there is a positive number  $\lambda > 0$  such that the weights  $W_n$ , eq.(16), are nonnegative for all paths  $\{z_j\}$  involving a number,  $n+1$ , of points that fulfils  $n > (t_b - t_a)/\lambda$ .*

The assumptions of the theorem about the potential may be written

$$\tilde{V}(q) = 0 \text{ for } |q| > R, \int_{-R}^R |\tilde{V}(q)| dq \leq K, \quad (17)$$

where  $K$  and  $R$  are positive parameters and  $\tilde{V}(q)$  is the Fourier transform of the potential, that is

$$\tilde{V}(q) \equiv \int V(x) \exp(ixq) dx.$$

Of course in a nonrelativistic theory the zero of energies may be fixed arbitrarily so that  $V = \text{const}$  is physically equivalent to  $V = 0$ . Thus the latter

eq.(17) should be understood modulo an appropriate redefinition of the zero of the potential.

Before going to the proof I will comment on the rationale for the hypotheses of the theorem. We are interested in the positivity of the weights  $W_n$  only in the limit  $n \rightarrow \infty$  (or  $\varepsilon \rightarrow 0$ ), and this is guaranteed if  $W_n$  is positive for any  $n$  large enough. On the other hand the constraints eq.(17) guarantee that the classical force is finite at any point, a rather obvious physical requirement. In fact the force on the particle at the point  $x$  is given by

$$\begin{aligned} F(x) &= -dV(x)/dx = -\frac{d}{dx} \left[ \frac{1}{2\pi} \int \tilde{V}(q) \exp(-ixq) dq \right] \\ &= -\frac{i}{2\pi} \int q \tilde{V}(q) \exp(-ixq) dq, \end{aligned}$$

where we have used the inverse Fourier transform of the latter eq.(17). Hence the force is bounded, that is

$$|F(x)| \leq \frac{1}{2\pi} \int_{-R}^R |q \tilde{V}(q)| dq \leq \frac{R}{2\pi} \int_{-R}^R |\tilde{V}(q)| dq \leq \frac{RK}{2\pi},$$

where the former eq.(17) has been taken into account. Thus all physically plausible potentials defined in a finite region of space are allowed by the constraints eq.(17) of our theorem if  $K$  and  $R$  are large enough. This includes the constant potential and the truncated harmonic oscillator

$$V(x) = \frac{1}{2}kx^2 \text{ if } |x| \leq L, 0 \text{ otherwise,}$$

with  $k$  not too large, that is  $k < K/(2RL^2)$ .

### Proof of the theorem

I start with a change leading to a description of the transition probability, different but equivalent to eq.(11) in the limit  $n \rightarrow \infty$ . I will write

$$\begin{aligned} P &= \lim_{n \rightarrow \infty} \left( \frac{1}{2\pi\varepsilon} \right)^n \int dz_{n-1} \dots \int dz_1 \int du_{n-1} \dots \int du_1 \\ &\quad \times \prod_{j=1}^{n-1} \exp[-\gamma|z_j| - \gamma|u_j|] \exp \left[ -\frac{i}{\varepsilon} u_j (z_{j-1} - 2z_j + z_{j+1}) \right] \\ &\quad \times \prod_{j=1}^{n-1} \left\{ 1 + i\varepsilon \left[ V(z_j - \frac{1}{2}z_j) - V(z_j + \frac{1}{2}z_j) \right] \right\}. \end{aligned} \quad (18)$$



The proof of equivalence follows from expanding the exponentials in the latter product of eq.(11) in powers of the small parameter  $\varepsilon$ , that is

$$\exp(i\varepsilon B_j) = 1 + i\varepsilon B_j - \frac{\varepsilon^2}{2} B_j^2 - \frac{i\varepsilon^3}{6} B_j^3 + \dots, \quad (19)$$

where for short I have labelled

$$B_j \equiv V(r_j - \frac{1}{2}u_j) - V(r_j + \frac{1}{2}u_j).$$

The relevant result is that only the term of order  $\varepsilon$  in eq.(19) contributes to eq.(18) in the limit  $\varepsilon \rightarrow 0$ . In fact we have

$$\prod_j (1 + i\varepsilon B_j) = 1 + i\varepsilon \sum_j B_j - \varepsilon^2 \sum_l \sum_{j>l} B_l B_j + \dots \quad (20)$$

As the sum  $\sum_j B_j$  consists of  $n - 1$  terms, the quantity  $\varepsilon \sum_j B_j$  has a finite limit when  $\varepsilon \rightarrow 0$  (remember that we assume  $n\varepsilon = t_b - t_a$ , finite in the limit). Similarly there are  $n(n - 1)/2$  terms in the double sum  $\sum_l \sum_{j>l} B_l B_j$  so that its product times  $\varepsilon^2$  has also a finite limit. The same happens for every term in the right side of eq.(20). In sharp contrast the terms containing  $B_j^s$  in eq.(19) with  $s > 1$  have extra factors  $\varepsilon$  whence they do not contribute in the limit  $\varepsilon \rightarrow 0$ . For instance there will be  $n(n - 1)$  terms of order  $\varepsilon^3$  not included in the sum eq.(20), namely those of the form

$$(i\varepsilon B_j) \left( -\frac{\varepsilon^2}{2} B_k^2 \right) = -\frac{i\varepsilon^3}{2} B_j B_k^2.$$

The sum of these terms contributes a quantity of order  $n^2 \varepsilon^3$  that will go to zero in the limit  $\varepsilon \rightarrow 0$ . A similar argument is valid for  $s > 3$ . This completes the proof that eq.(18) is equivalent to eq.(13).

A sufficient condition for the nonnegativity of  $W_n$ , eq.(16) is that  $Q_j \geq 0$  for any  $j$ , with  $Q_j$  now redefined as

$$\begin{aligned} Q_j &= (2\pi\varepsilon)^{-1} \exp[-\gamma |z_j|] \int du_j \exp[-\gamma |u_j| - iu_j s_j] \\ &\times \left\{ 1 + i\varepsilon \left[ V(z_j - \frac{1}{2}u_j) - V(z_j + \frac{1}{2}u_j) \right] \right\}, \end{aligned} \quad (21)$$

where  $s_j$  was defined in eq.(12). The potential may be written in terms of its Fourier transform, which leads to

$$V(z_j - \frac{1}{2}u_j) - V(z_j + \frac{1}{2}u_j) = \frac{1}{2\pi} \int_{-R}^R dq_j \left| \tilde{V}(q_j) \right| \exp(-iz_j q_j) \times [\exp(iu_j q_j) - \exp(-iu_j q_j)],$$

where I have taken into account the constraint eq.(17). If this is inserted in eq.(21) the  $u_j$  integrals are trivial and we get

$$\begin{aligned} Q_j &= (2\pi\varepsilon)^{-1} \exp[-\gamma|z_j|] \frac{2\gamma}{s_j^2 + \gamma^2} (1 - \varepsilon M_j), \\ M_j &= \pi^{-1} \int_0^R dq_j \operatorname{Im} \left[ \tilde{V}(q_j) \exp(-iz_j q_j) \right] \\ &\quad \times \left[ \frac{s_j^2 + \gamma^2}{(s_j - q_j)^2 + \gamma^2} - \frac{s_j^2 + \gamma^2}{(s_j + q_j)^2 + \gamma^2} \right], \end{aligned} \quad (22)$$

where I have taken into account that  $\tilde{V}(-q_j) = \tilde{V}^*(q_j)$  so that the real part of  $\tilde{V}(q_j) \exp(-iz_j q_j)$  does not contribute, and the imaginary part is odd with respect to  $q_j$ .

Now  $Q_j$  would be nonnegative if  $M_j \leq 1/\varepsilon$ , and this will happen for any  $\varepsilon \leq \lambda$  fulfilling  $1/\lambda \geq M_j(z_j, s_j)$  for all  $j$ . That is the parameter  $\lambda$  proposed in the theorem exists if  $M_j$  is bounded from above. In fact there is a bound that may be calculated as follows. Taking into account that  $\gamma \ll R$  and  $|q_j| \leq R$ , we have

$$\left| \frac{s_j^2 + \gamma^2}{(s_j - q_j)^2 + \gamma^2} - \frac{s_j^2 + \gamma^2}{(s_j + q_j)^2 + \gamma^2} \right| \lesssim \frac{R^2}{\gamma^2},$$

the maximum corresponding to  $|s_j| = |q_j| = R$ . This leads to

$$M_j \leq |M_j| \leq \frac{1}{2\pi} \int_{-R}^R dq_j \left| \tilde{V}(q_j) \right| \frac{R^2}{\gamma^2} \leq \frac{R^2 K}{2\pi\gamma^2}, \quad (23)$$

where the constraints eqs.(17) have been taken into account. As a consequence there is a parameter  $\lambda = 2\pi\gamma^2/(R^2 K) > 0$  as stated, that *completes the proof of the theorem*.

As a consequence of the theorem the quantity  $W_n(\{z_j\})$  eq.(16) is non-negative in the limit  $n \rightarrow \infty$  for all values of the (real positive) parameter  $\gamma$ .

Thus  $W_n(\{z_j\})$  might be interpreted as the probability of a path defined by the spacetime points  $(z_j, t_j)$ . When  $\gamma$  approaches 0 the quantity  $P(a \rightarrow b)$ , eq.(13), approaches the quantum transition probability, that is the square of the quantum transition amplitude defined in eq.(5). Thus the quantum transition probability may be obtained in the form of a functional integral over particle paths, that may be described as follows

$$P(a \rightarrow b) = \frac{1}{2\pi(t_b - t_a)} \lim_{\gamma \rightarrow 0} \lim_{\varepsilon \rightarrow 0} n \int dz_1 \dots \int dz_{n-1} \prod_{j=1}^{n-1} Q_j, \quad (24)$$

where  $Q_j$  was given in eq.(21). It is necessary that the limit  $\gamma \rightarrow 0$  is taken after the limit  $\varepsilon \rightarrow 0$  in order that the  $z_j$  integrals are well defined.

The calculations using eq.(24) are rather involved even for simple potentials. For instance for the free particle, that is  $V(x) = 0$ , the integral in  $z_1$  involves the two functions  $Q_1$  and  $Q_2$  because  $z_1$  enters in both, that is

$$P_{01} \equiv \int_{-\infty}^{\infty} Q_1 Q_2 dz_1 = 2(2\pi\varepsilon)^{-2} \int_0^{\infty} \exp(-\gamma z_1) \times \frac{4\gamma^2 dz_1}{\{[(z_2 - 2z_1 + z_a)/\varepsilon]^2 + \gamma^2\} \{[(z_3 - 2z_2 + z_1)/\varepsilon]^2 + \gamma^2\}}. \quad (25)$$

After this integral is performed we could make the integral in  $z_2$  and so on. The calculation is straightforward but lengthy and it will not be continued here. However the physics of eq.(25) is clear, when  $\gamma$  is small the function under the integral is picked at

$$z_2 - 2z_1 + r_a \simeq z_3 - 2z_2 + z_1 \simeq 0 \Rightarrow z_3 - z_2 \simeq z_2 - z_1 \simeq z_1 - r_a.$$

That is the velocity changes but slightly whence the motion becomes close to classical. Therefore in the formalism the quantum free particle follows the classical path with probability 1 in the limit  $\gamma \rightarrow 0$ . For other potentials,  $V(x) \neq \text{const}$ , there is some probability that the velocity changes and the probability depends, via the Fourier transform of the potential, on some region around the instantaneous position of the particle. That is *the potential produces an effect which is stochastic and nonlocal*.

From the mathematical point of view  $W_n(\{z_j\})$  is a stochastic process with discrete time for finite  $n$ , that is a stochastic chain and becomes continuous in the limit  $n \rightarrow \infty$  (with  $n\varepsilon = t_b - t_a$ ). The chain (or the continuous)

stochastic process is not Markovian, as shown in eq.(25) where the transition probability from  $z_0$  to  $z_1$  depends not only on these two values but also in  $z_2$  and  $z_3$ . This contrasts with the path integral formulation of the quantum amplitude, eq.(6), where every term in the product depends only on the two positions involved,  $x_j$  and  $x_{j-1}$ . Therefore the Chapman-Kolmogorov equation is not fulfilled, that is the equality

$$\int dr_c P(r_a, t_a \rightarrow r_c, t_c) P(r_c, t_c \rightarrow r_b, t_b) = P(r_a, t_a \rightarrow r_b, t_b) \quad (26)$$

does not hold true in general. This is the reason why it is not possible to get the quantum transition probability as the solution of a Fokker-Planck equation. In contrast the quantum amplitudes do fulfil an equation similar to eq.(26), although involving complex amplitudes rather than real positive probabilities, that is

$$\int dr_c A(r_a, t_a \rightarrow r_c, t_c) A(r_c, t_c \rightarrow r_b, t_b) = A(r_a, t_a \rightarrow r_b, t_b),$$

This is consistent with the amplitude being governed by an equation resembling a diffusion (Fokker-Planck) equation although in complex space, that is Schrödinger eq.(1).

## IV. Conclusions

The formalism developed in this paper provides a description of (non-relativistic) quantum motion more detailed than the standard quantum description, but in agreement with QM when appropriate averages are made. In particular, if we add the probabilities of different paths in order to get the total transition probability, see eq.(15). Thus the formalism is a hidden variables theory (HVT) of quantum mechanics. The theory is stochastic in the sense that it does not provide a deterministic law of motion (as for instance Bohm's HVT does) but the probability of the different possible paths of the quantum particle.

A path may be defined by the positions  $\{r_a \equiv r_0, \dots, r_j, \dots, r_b \equiv r_n\}$  at times  $t_a, \dots, t_j \equiv t_a + j\varepsilon, \dots, t_b$  or, what is equivalent, the end positions plus the velocity changes  $\{s_j\}$  at times  $t_j$ . Eventually we should consider the limit  $n \rightarrow \infty$  with  $n\varepsilon = t_b - t_a$ . These probabilities depend on the potential  $V(r)$  along the path, therefore we may say that the potential governs the motion of the

particle, as is shown by eqs.(??) and (??) . However the action of the potential is less direct than in the classical case, in particular the action depends, via the Fourier transform of the potential, on a whole region around the particle. Thus the hidden variables model is nonlocal.

I emphasize again that we remain at the level of non-relativistic quantum mechanics. I do not claim that a similar interpretation may be extended to relativistic quantum field theory, e. g. photons, electrons when spin plays a role, or even atoms or molecules when (Bose or Fermi) statistics is relevant. The existence of pictures for physical theories is considered irrelevant, even useless, for many people. But for some scientists “pictures of the reality”<sup>1</sup> are an essential part of physics. Also the HVT may be useful for some calculations where we want to emphasize the particle aspect of quantum systems, like molecular dynamics.

## References

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